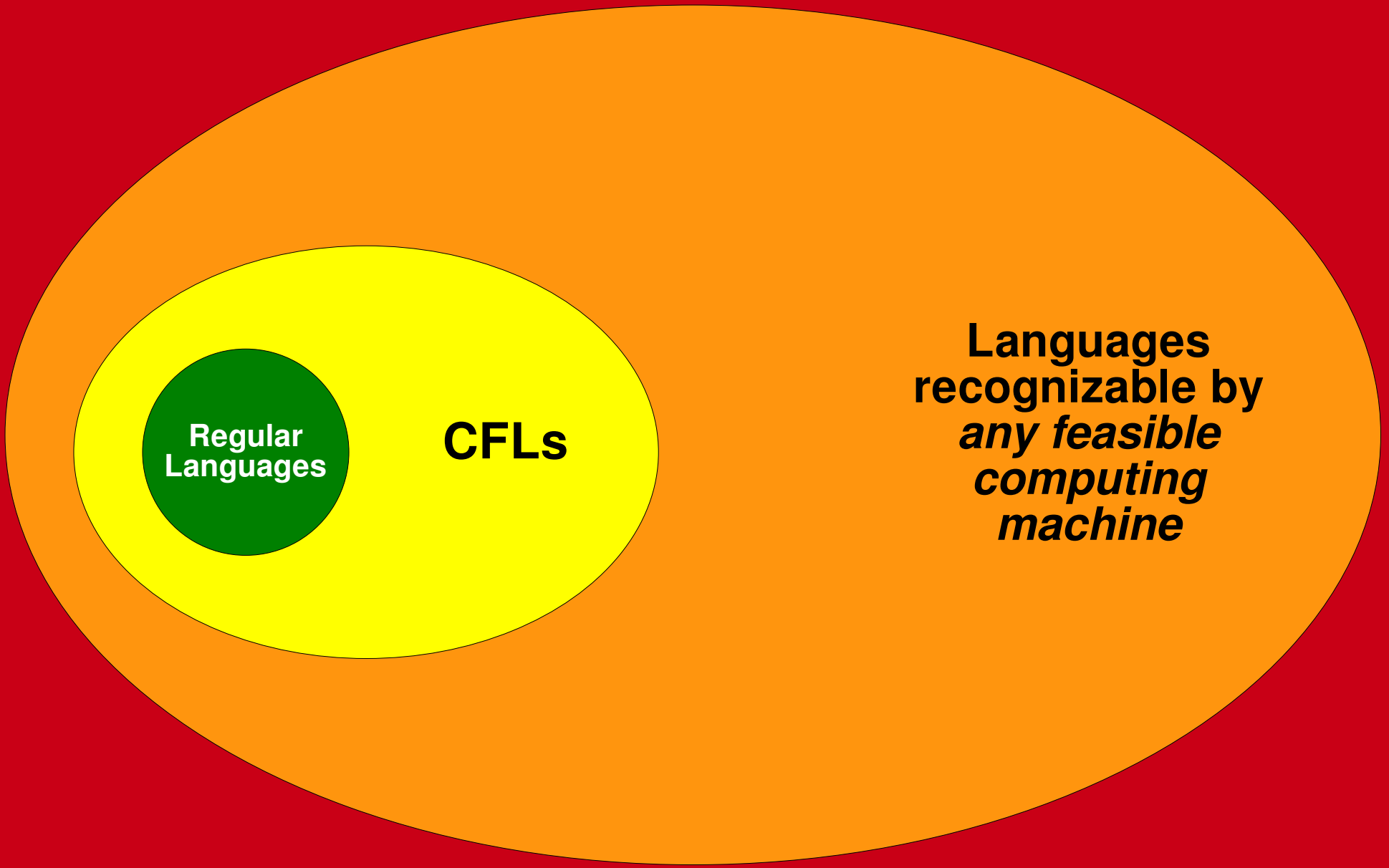
CS103
FALL 2025

Lecture 20:

# Turing Machines

## Part 1 of 3

What problems can we solve with a computer?


Regular Languages
CFLs
Languages recognizable by *any feasible computing machine*
All Languages

# That same drawing, to scale.

**All Languages**

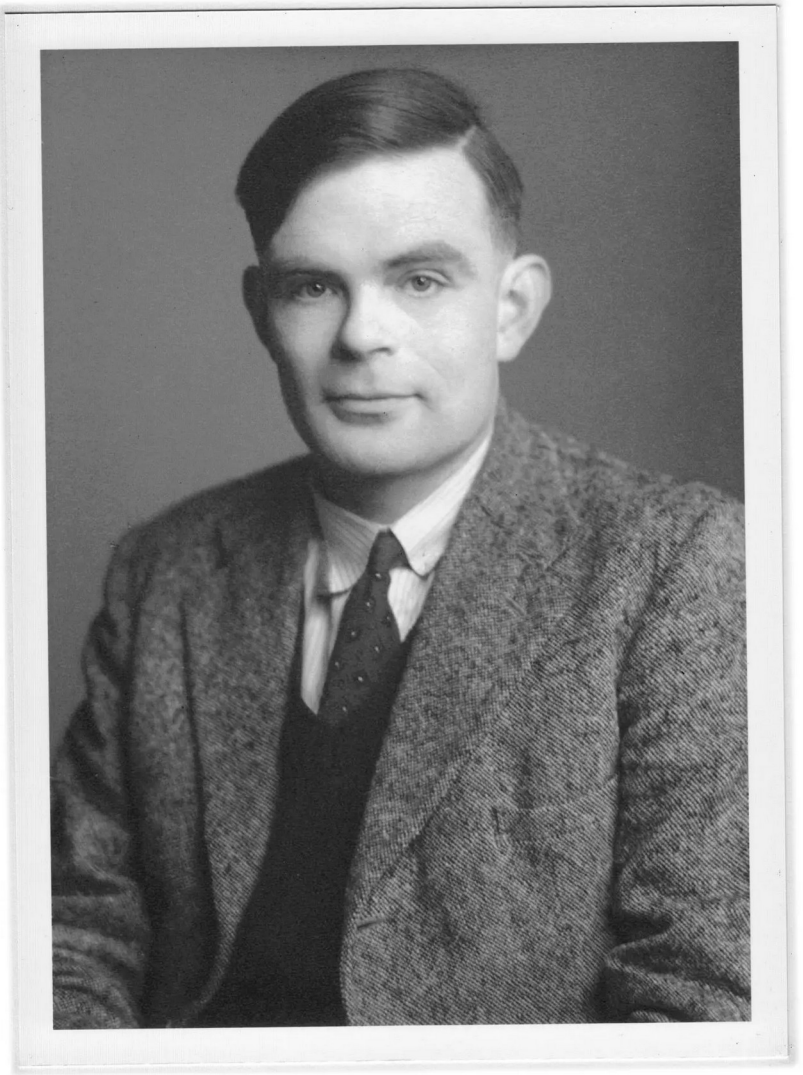
# The Problem

- Finite automata accept precisely the regular languages.
- We may need unbounded memory to recognize context-free languages.
  - e.g. $\{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$ requires unbounded counting.
- How do we model a computing device that has unbounded memory?

# A Brief History Lesson

# Turing Machines

- In March 1936, Alan Turing (aged 23!) published a paper detailing the ***a-machine*** (for ***automatic machine***), an automaton for computing on real numbers.
- They're now more popularly referred to as ***Turing machines*** in his honor.
- He also later made contributions to computational biology, artificial intelligence, cryptography, etc. Seriously, Google this guy.



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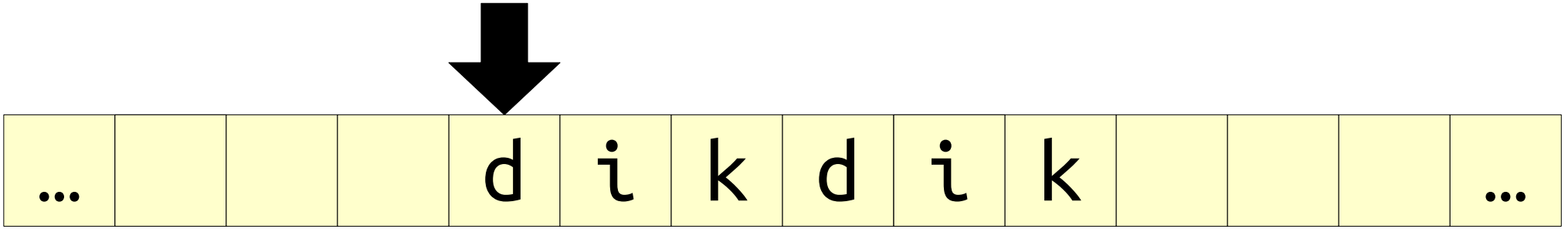
***Key Idea:*** Even if you need huge amounts of scratch space to perform a calculation, at each point in the calculation you only need access to a small amount of that scratch space.

# Turing Machines

- To provide his machines extra memory, Turing gave his machines access to an ***infinite tape*** subdivided into a number of ***tape cells***.
- A Turing machine can only see one tape cell at a time, the one pointed at by the ***tape head***.
- The Turing machine can
  - read the cell under the tape head,
  - (possibly) change which symbol was written under the tape head, and
  - move its tape head to the left or to the right.



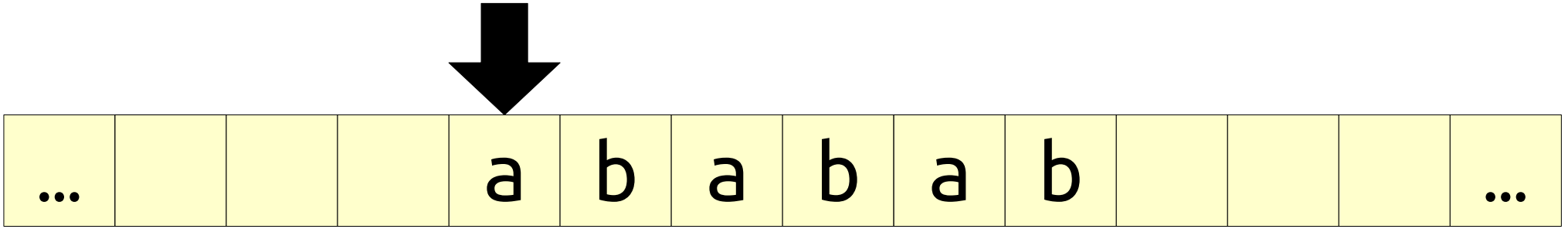
# Turing Machines

- Over the years, there have been many simplifications and edits to Turing's original automata.
  - In practice, electronic computers are written in terms of individual instructions rather than states and transitions.
  - Turing's original paper deals with computing individual real numbers; we typically want to compute functions of inputs.
- What we're going to present as "Turing machines" in this class differ significantly from Turing's original description, while retaining the core essential ideas.
  - (Our model is closer to Emil Post's *Formulation 1* and Hao Wang's *Basic Machine B*, for those of you who are curious.)
- If you'd like to learn more about Turing's original version of the Turing machine, come chat with us after class!

# Turing Machines

- A TM is a series of instructions that control a tape head as it moves across an infinite tape.
- The tape begins with the input string written somewhere, surrounded by infinitely many blank cells.
  - Rule: The input string cannot contain blank cells.
- The tape head begins above the first character of the input. (If the input is ε, the tape head points somewhere on a blank tape.)

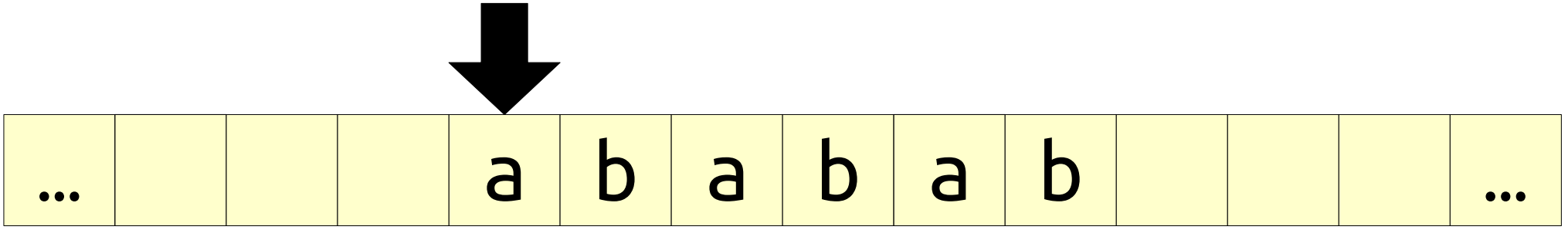
```
Start:
  If Blank Return True
  If 'b' Return False
  Write 'x'
  Move Right
  If Not 'b' Return False
  Write 'x'
  Move Right
  Goto Start
```



# Turing Machines

- We begin at the `Start` label.
- Labels indicate different sections of code. The name `Start` is special and means “begin here.”
- Labels have no effect when executed. We just move to the next line.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



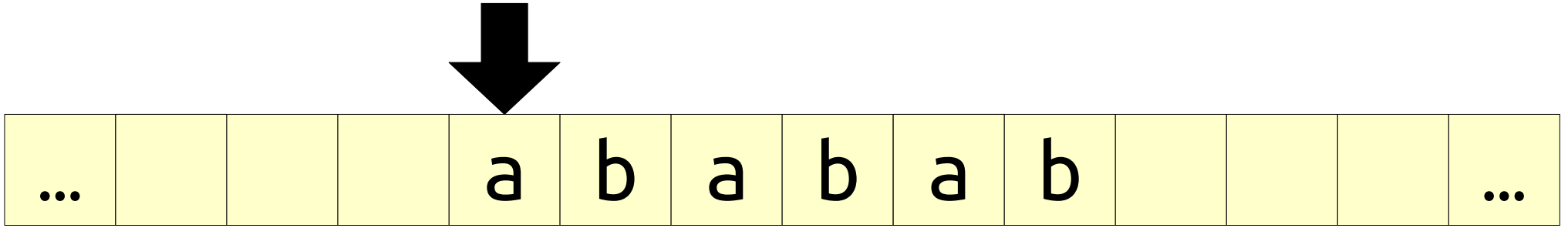
# Turing Machines

- A statement of the form

  **If** *symbol command*

  checks if the character under the tape head is *symbol*.
- If so, it executes *command*.
- If not, nothing happens.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



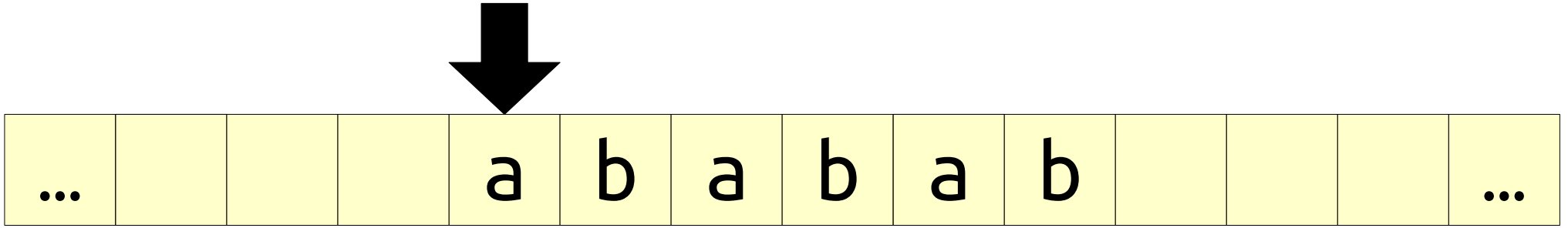
# Turing Machines

- The statement

  **`Write`** *symbol*

  writes *symbol* to the cell under the tape head.
- The *symbol* can either be `Blank` or a character in quotes.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



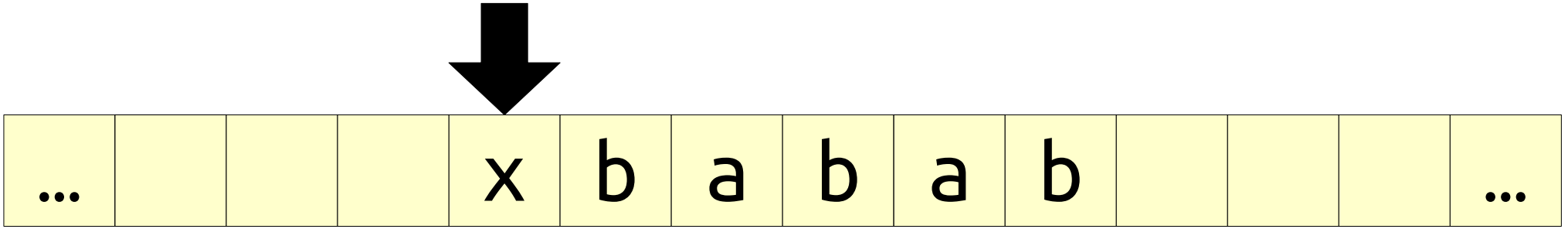
# Turing Machines

- The command

  **Move** *direction*

  moves the tape head one step in the indicated direction (either `Left` or `Right`).

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



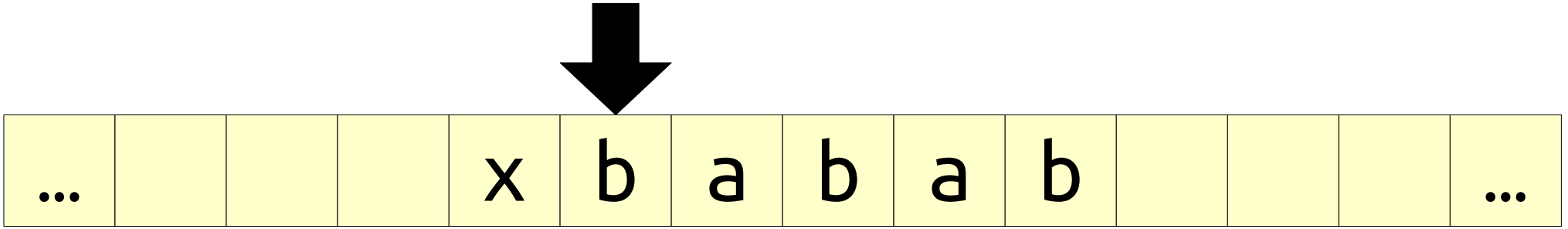
# Turing Machines

- A statement of the form

  **`If Not`** *symbol command*

  sees if the cell under the tape head holds *symbol.*
- If so, nothing happens.
- If not, it executes *command*.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



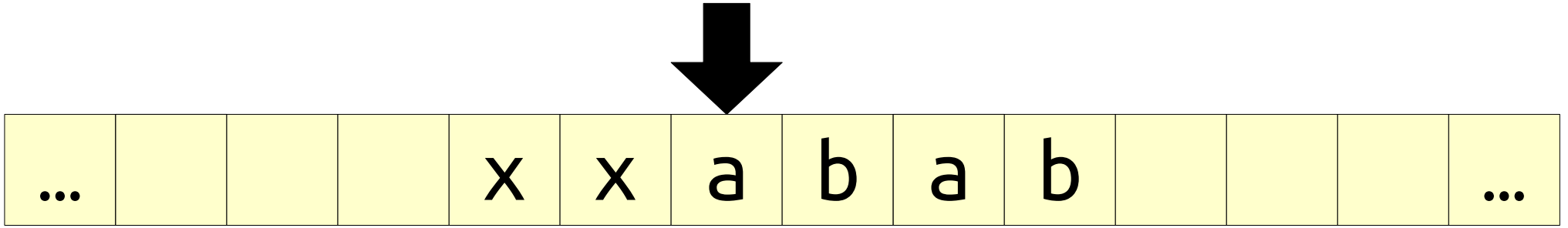
# Turing Machines

- The command

  **`Goto`** *label*

  jumps to the indicated label.
- This program just has a `Start` label, but most interesting programs have other labels beyond this.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



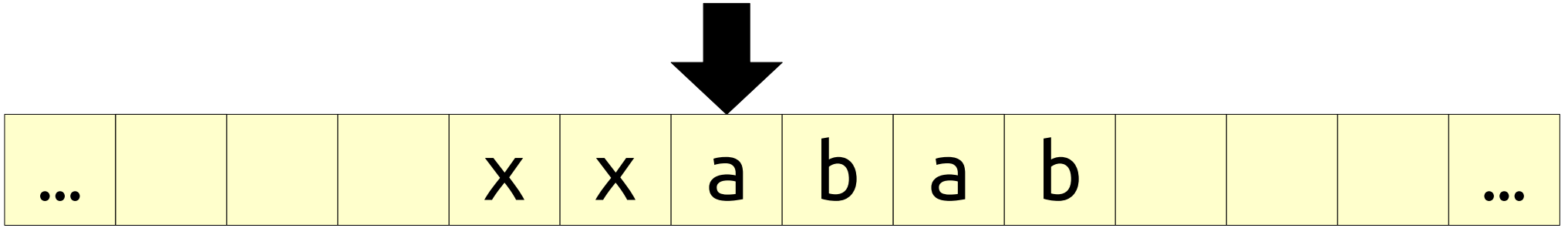
# Turing Machines

- A TM stops when executing the

  **`Return`** *result*

  command.
- Here, *result* can be either `True` or `False`.
- (If we “fall off” the bottom of the program, the TM acts as though it executes the `Return False` command.)

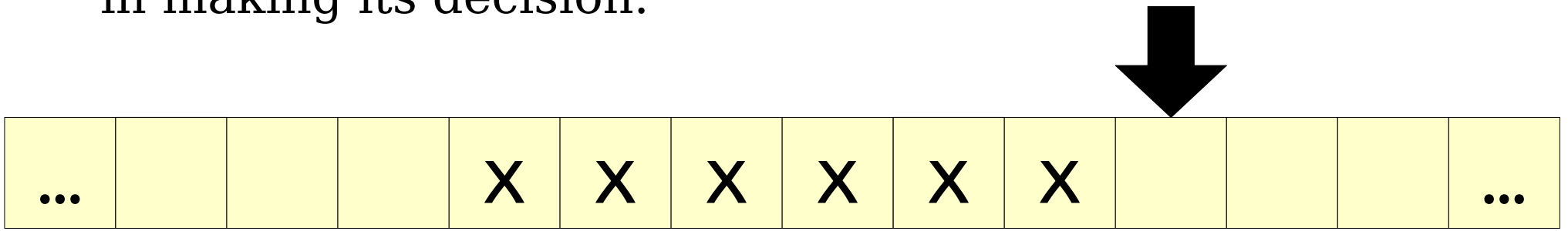
```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



# Turing Machines

- This TM initially started up with the string ababab on its tape, so this means that TM returns true on the input ababab, not xxxxxx.
- An intuition for this: we gave this program an input. It therefore returned true with respect to that input, not whatever internal data it generated in making its decision.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



# Turing Machines

- To summarize, we only have six commands:
  - `Move` *direction*
  - `Write` *symbol*
  - `Goto` *label*
  - `Return` *result*
  - `If` *symbol command*
  - `If Not` *symbol command*
- Despite their simplicitly, TMs are *surprisingly* powerful. The rest of this lecture explores why.

```
Start:
  If Blank Return True
  If 'b' Return False
  Write 'x'
  Move Right
  If Not 'b' Return False
  Write 'x'
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  Goto Start
```
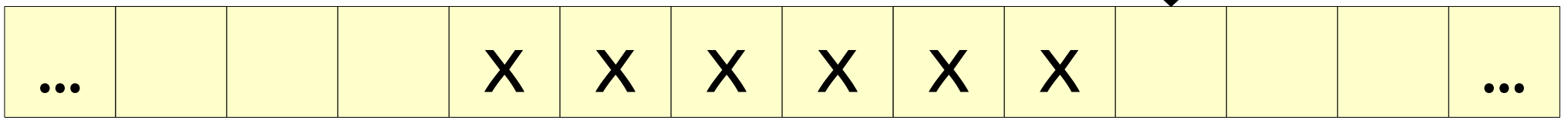



# Programming Turing Machines

# Our First Challenge

- The language

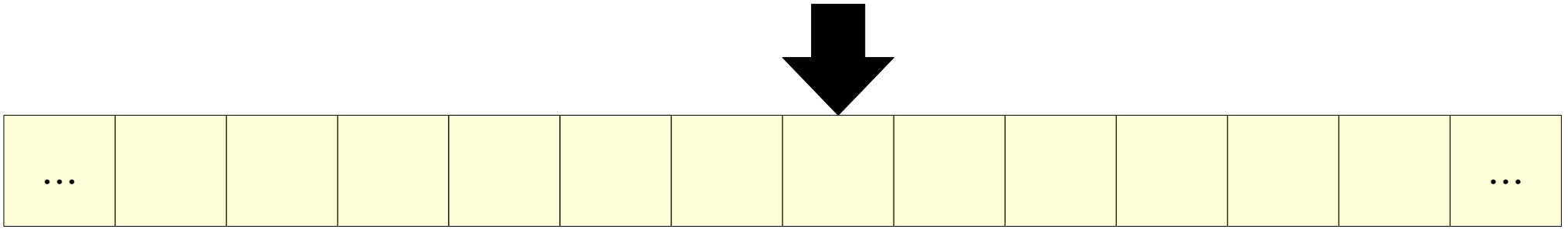
$$\{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$$

  is a canonical example of a nonregular language. It's not possible to check if a string is in this language given only finite memory.
- Turing machines, however, are powerful enough to do this. Let's see how.

# A Recursive Approach

- We can process our string using this recursive approach:
  - The string ε is in $L$.
  - The string **a**$w$**b** is in $L$ if and only if $w$ is in $L$.
  - Any string starting with **b** is not in $L$.
  - Any string ending with **a** is not in $L$.
- All that's left to do now is write a TM that implements this.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write Blank

ZipRight:
   Move Right
   If Not Blank Goto ZipRight
   Move Left
   If Not 'b' Return False
   Write Blank

ZipLeft:
   Move Left
   If Not Blank Goto ZipLeft
   Move Right
   Goto Start
```

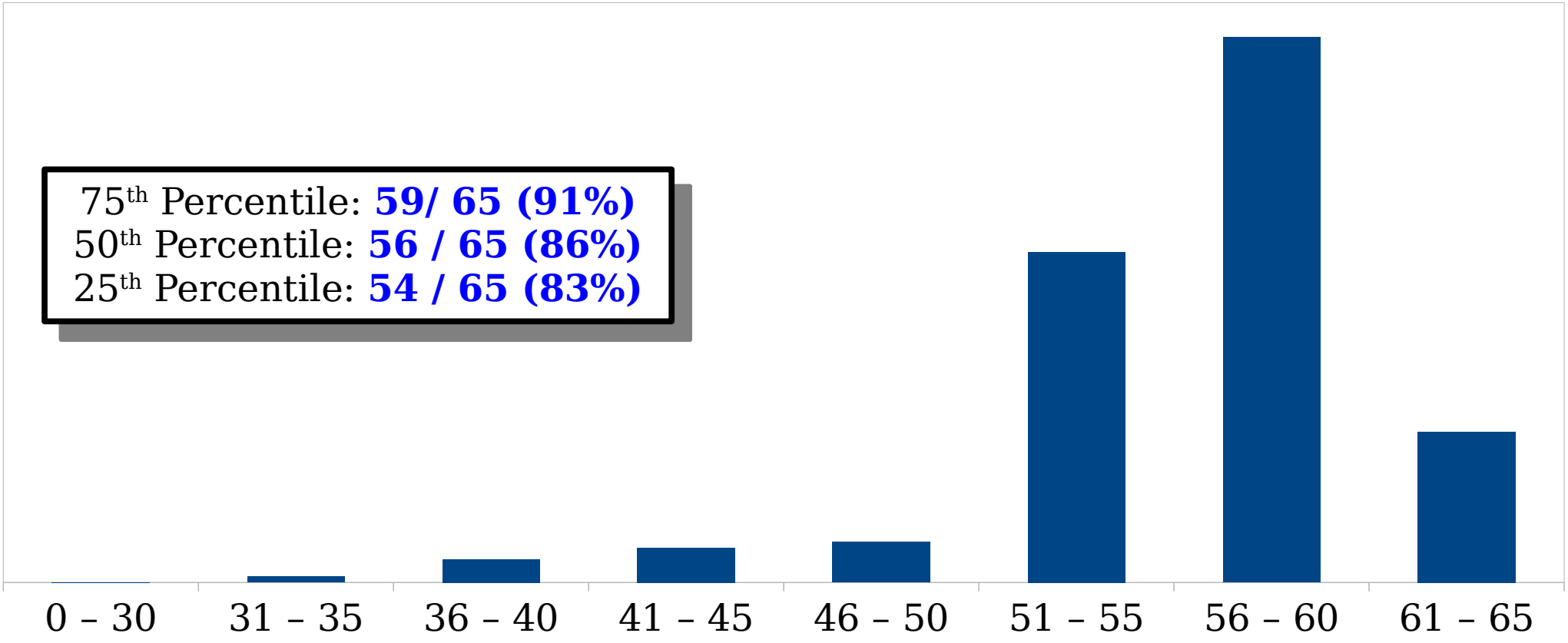


Time-Out for Announcements!

# Midterm Exam

- You're done with the second midterm! Hooray!
- We will be grading the exam this weekend and will release scores as soon as they're ready.
- We've released preliminary solutions to the exam on the course website.
  - We don't have statistics, common mistakes, etc., and only have one solution route for each proof.
- ***Do not change your grading basis or withdraw*** until you've run some grade calculations; use Sean's amazing ***Grade Cruncher*** on the course website.

# Problem Set 6 Graded



Back to CS103!

# Our Next Challenge

- Let's now take aim at this more general language:

$$\{ w \in \{\texttt{a}, \texttt{b}\}^* \mid w \text{ has an equal number of } \texttt{a}\text{'s and } \texttt{b}\text{'s } \}$$

- This language is not regular (do you see why?)
- It is context-free, but it's a bit tricky to write a CFG for it. (This is a great exercise!)
- Let's see how to design a TM for it.

# Another Idea

- We just built a TM for the language

  $$\{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ has the same number of } \mathtt{a}\text{'s and } \mathtt{b}\text{'s} \}.$$

- An observation: this would be a *lot* easier to test for if all the **a**'s came before all the **b**'s.
  - In fact, that would turn this into checking if the string has the form $\mathtt{a}^n\mathtt{b}^n$, which we already know how to do!
- ***Idea:*** Could we sort the characters of our input string?

# Exploring This Idea

# Cool TM Tricks 1: ***Fibonacci Numbers***

# Cool TM Tricks 2: ***Decimal Fibonacci***

# Summary for Today

- Turing machines are abstract computers that issue commands to an infinite tape subdivided into cells.
- Each step of the TM can move the tape head, change what's on the tape, or jump to a different part of the program.
- TMs can be composed together to build larger TMs out of smaller ones.

# Next Time

- ***The Church-Turing Thesis***
  - How powerful are Turing machines?
- ***Decidability and Recognizability***
  - Two notions of “solving a problem.”